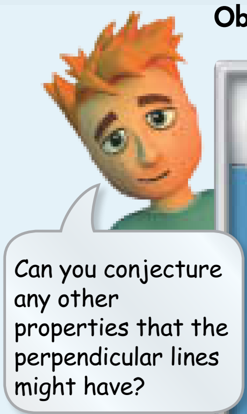
5-3 Bisectors in Triangles

Common Core State Standards
G-C.A.3 Construct the inscribed and circumscribed circles of a triangle . . .
MP 1, MP 3, MP 4, MP 7, MP 8

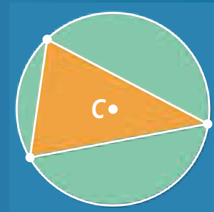
Objective To identify properties of perpendicular bisectors and angle bisectors



MATHEMATICAL PRACTICES

SOLVE IT! Getting Ready!

Construct a circle and label its center C. Choose any three points on the circle and connect them to form a triangle. Draw three lines from C such that each line is perpendicular to one side of the triangle. What conjecture can you make about the two segments into which each side of the triangle is divided? Justify your reasoning.



In the Solve It, the three lines you drew intersect at one point, the center of the circle. When three or more lines intersect at one point, they are **concurrent.** The point at which they intersect is the **point of concurrency.**

Essential Understanding For any triangle, certain sets of lines are always concurrent. Two of these sets of lines are the perpendicular bisectors of the triangle's three sides and the bisectors of the triangle's three angles.

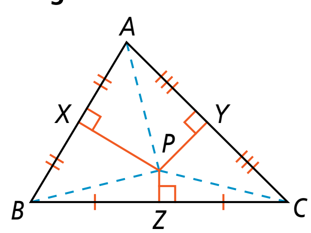
Lesson Vocabulary
- concurrent
- point of concurrency
- circumcenter of a triangle
- circumscribed about
- incenter of a triangle
- inscribed in

take note

Theorem 5-6 Concurrency of Perpendicular Bisectors Theorem

Theorem
The perpendicular bisectors of the sides of a triangle are concurrent at a point equidistant from the vertices.

Diagram

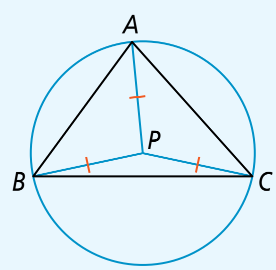


Symbols
Perpendicular bisectors $\overline{PX}$, $\overline{PY}$, and $\overline{PZ}$ are concurrent at P.

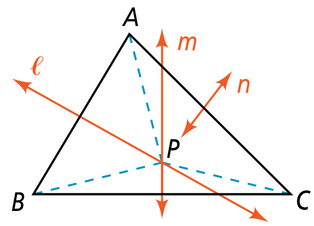
$PA = PB = PC$

The point of concurrency of the perpendicular bisectors of a triangle is called the **circumcenter of the triangle.**

Since the circumcenter is equidistant from the vertices, you can use the circumcenter as the center of the circle that contains each vertex of the triangle. You say the circle is **circumscribed about** the triangle.



Proof Proof of Theorem 5-6



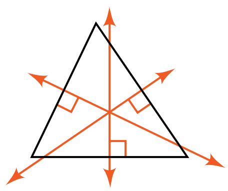
Given: Lines ℓ, m, and n are the perpendicular bisectors of the sides of $\triangle ABC$. P is the intersection of lines ℓ and m.

Prove: Line n contains point P, and $PA = PB = PC$.

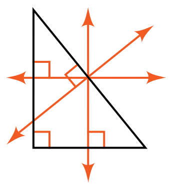
Proof: A point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment. Point P is on ℓ, which is the perpendicular bisector of $\overline{AB}$, so $PA = PB$. Using the same reasoning, since P is on m, and m is the perpendicular bisector of $\overline{BC}$, $PB = PC$. Thus, $PA = PC$ by the Transitive Property. Since $PA = PC$, P is equidistant from the endpoints of $\overline{AC}$. Then, by the converse of the Perpendicular Bisector Theorem, P is on line n, the perpendicular bisector of $\overline{AC}$.

The circumcenter of a triangle can be inside, on, or outside a triangle.

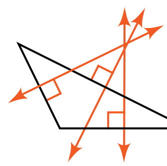
Acute triangle



Right triangle



Obtuse triangle

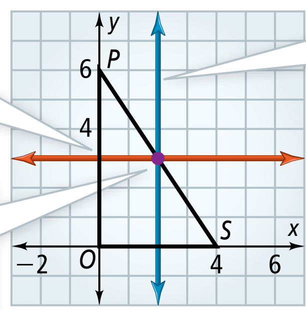


Problem 1 Finding the Circumcenter of a Triangle

What are the coordinates of the circumcenter of the triangle with vertices $P(0, 6)$, $O(0, 0)$, and $S(4, 0)$?

Find the intersection point of two of the triangle's perpendicular bisectors. Here, it is easiest to find the perpendicular bisectors of $\overline{PO}$ and $\overline{OS}$.

Step 1 (0, 3) is the midpoint of $\overline{PO}$. The line through (0, 3) that is perpendicular to $\overline{PO}$ is $y = 3$.



Step 2 (2, 0) is the midpoint of $\overline{OS}$. The line through (2, 0) that is perpendicular to $\overline{OS}$ is $x = 2$.

Step 3 Find the point where the two perpendicular bisectors intersect. $x = 2$ and $y = 3$ intersect at (2, 3).

Think
Does the location of the circumcenter make sense?
Yes, $\triangle POS$ is a right triangle, so its circumcenter should lie on its hypotenuse.

The coordinates of the circumcenter of the triangle are (2, 3).

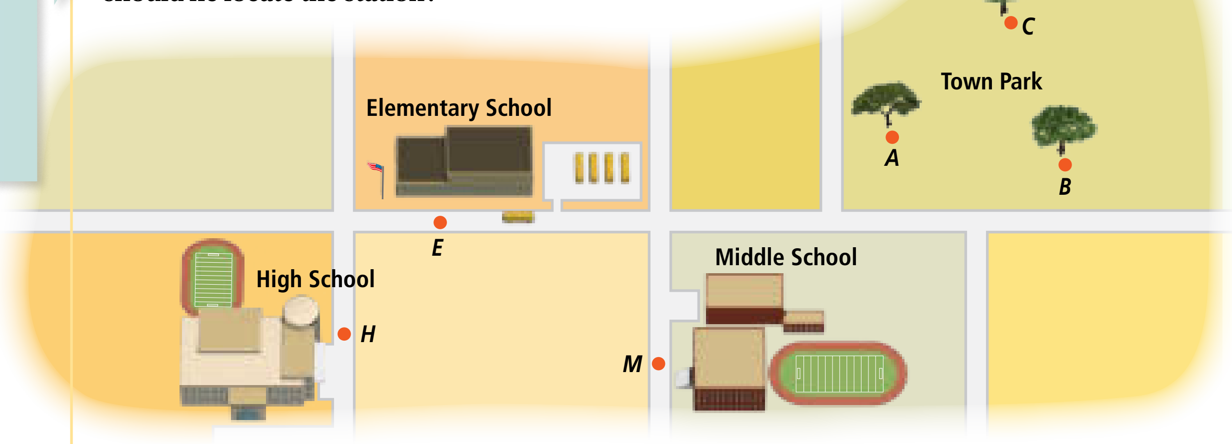
Got It? 1. What are the coordinates of the circumcenter of the triangle with vertices $A(2, 7)$, $B(10, 7)$, and $C(10, 3)$?

Think

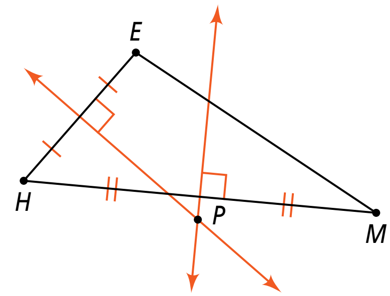
How do you find a point equidistant from three points?
As long as the three points are noncollinear, they are vertices of a triangle. Find the circumcenter of the triangle.

Problem 2 Using a Circumcenter

A town planner wants to locate a new fire station equidistant from the elementary, middle, and high schools. Where should he locate the station?



The three schools form the vertices of a triangle. The planner should locate the fire station at P, the point of concurrency of the perpendicular bisectors of $\triangle EMH$. This point is the circumcenter of $\triangle EMH$ and is equidistant from the three schools at E, M, and H.



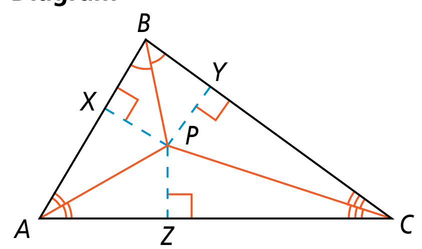
Got It? 2. In Problem 2, the town planner wants to place a bench equidistant from the three trees in the park. Where should he place the bench?

take note

Theorem 5-7 Concurrency of Angle Bisectors Theorem

Theorem
The bisectors of the angles of a triangle are concurrent at a point equidistant from the sides of the triangle.

Diagram

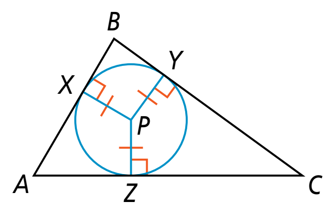


Symbols
Angle bisectors $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ are concurrent at P.

$PX = PY = PZ$

You will prove Theorem 5-7 in Exercise 24.

The point of concurrency of the angle bisectors of a triangle is called the **incenter of the triangle.** For any triangle, the incenter is always inside the triangle. In the diagram, points X, Y, and Z are equidistant from P, the incenter of $\triangle ABC$. P is the center of the circle that is **inscribed in** the triangle.

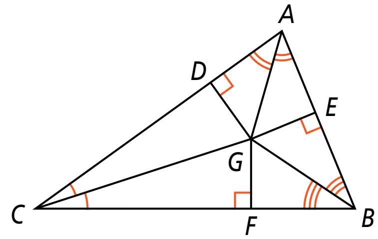


Problem 3 Identifying and Using the Incenter of a Triangle

Think

What is the distance from a point to a line?
The distance from a point to a line is the length of the perpendicular segment that joins the point to the line.

Algebra $GE = 2x - 7$ and $GF = x + 4$. What is GD?



G is the incenter of $\triangle ABC$ because it is the point of concurrency of the angle bisectors. By the Concurrency of Angle Bisectors Theorem, the distances from the incenter to the three sides of the triangle are equal, so $GE = GF = GD$. Use this relationship to find x.

$2x - 7 = x + 4$	$GE = GF$
$2x = x + 11$	Add 7 to each side.
$x = 11$	Subtract x from each side.

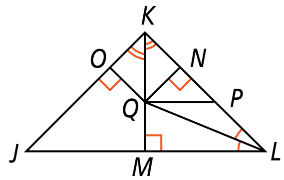
Now find GF.

$GF = x + 4$	
$= 11 + 4 = 15$	Substitute 11 for x.

Since $GF = GD$, $GD = 15$.

Got It? **3. a.** $QN = 5x + 36$ and $QM = 2x + 51$. What is QO?

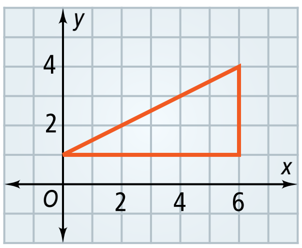
b. Reasoning Is it possible for QP to equal 50? Explain.



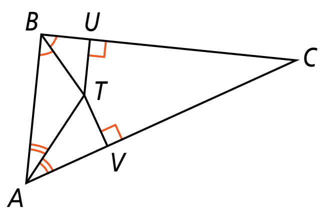
Lesson Check

Do you know HOW?

1. What are the coordinates of the circumcenter of the following triangle?



2. In the figure at the right, $TV = 3x - 12$ and $TU = 5x - 24$. What is the value of x?

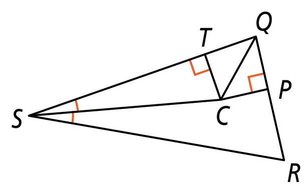


Do you UNDERSTAND? MATHEMATICAL PRACTICES

3. Vocabulary A triangle's circumcenter is outside the triangle. What type of triangle is it?

4. Reasoning You want to find the circumcenter of a triangle. Why do you only need to find the intersection of two of the triangle's perpendicular bisectors, instead of all three?

5. Error Analysis Your friend sees the triangle at the right and concludes that $CT = CP$. What is the error in your friend's reasoning?



6. Compare and Contrast How are the circumcenter and incenter of a triangle alike? How are they different?

Practice and Problem-Solving Exercises

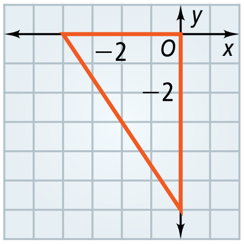
MATHEMATICAL PRACTICES

A Practice

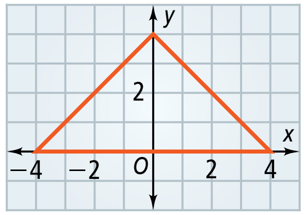
Coordinate Geometry Find the coordinates of the circumcenter of each triangle.

See Problem 1.

7.



8.



Coordinate Geometry Find the coordinates of the circumcenter of $\triangle ABC$.

9. $A(0, 0)$ $B(3, 0)$ $C(3, 2)$

10. $A(0, 0)$ $B(4, 0)$ $C(4, -3)$

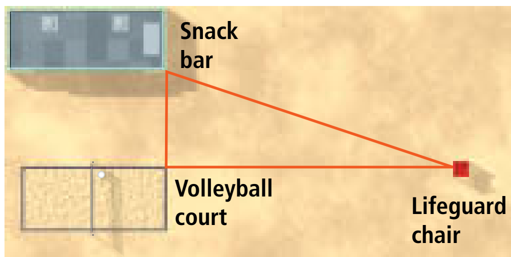
11. $A(-4, 5)$ $B(-2, 5)$ $C(-2, -2)$

12. $A(-1, -2)$ $B(-5, -2)$ $C(-1, -7)$

13. $A(1, 4)$ $B(1, 2)$ $C(6, 2)$

14. **City Planning** Copy the diagram of the beach. Show where town officials should place a recycling barrel so that it is equidistant from the lifeguard chair, the snack bar, and the volleyball court. Explain.

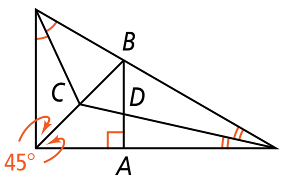
See Problem 2.



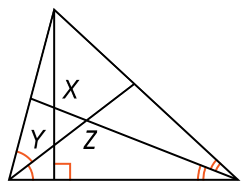
Name the point of concurrency of the angle bisectors.

See Problem 3.

15.

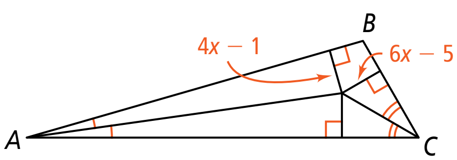


16.

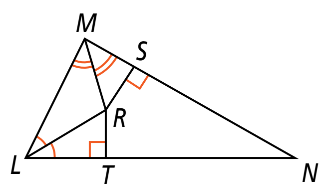


Find the value of *x*.

17.



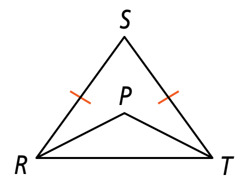
18. $RS = 4(x - 3) + 6$ and $RT = 5(2x - 6)$.



B Apply

19. Think About a Plan In the figure at the right, P is the incenter of isosceles $\triangle RST$. What type of triangle is $\triangle RPT$? Explain.

- What segments determine the incenter of a triangle?
- What do you know about the base angles of an isosceles triangle?

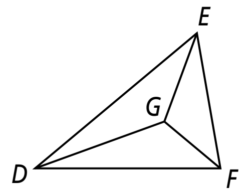


Constructions **Draw a triangle that fits the given description. Then construct the inscribed circle and the circumscribed circle. Describe your method.**

20. right triangle, $\triangle DEF$

21. obtuse triangle, $\triangle STU$

22. Algebra In the diagram at the right, G is the incenter of $\triangle DEF$, $m\angle DEF = 60$, and $m\angle EFD = 2 \cdot m\angle EDF$. What are $m\angle DGE$, $m\angle DGF$, and $m\angle EGF$?



23. Writing Ivars found an old piece of paper inside an antique book.
It read,

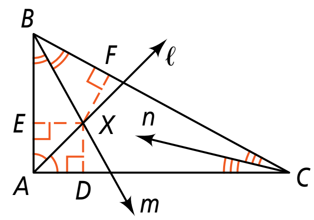
From the spot I buried Olaf's treasure, equal sets of paces did I measure; each of three directions in a line, there to plant a seedling Norway pine. I could not return for failing health; now the hounds of Haiti guard my wealth. —Karl

After searching Caribbean islands for five years, Ivars found an island with three tall Norway pines. How might Ivars find where Karl buried Olaf's treasure?

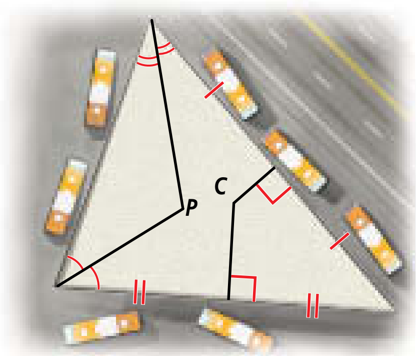
24. Use the diagram at the right to prove the Concurrrency of Angle Bisectors Theorem.
Proof

Given: Rays ℓ, m, and n are bisectors of the angles of $\triangle ABC$. X is the intersection of rays ℓ and m, $\overline{XD} \perp \overline{AC}$, $\overline{XE} \perp \overline{AB}$, and $\overline{XF} \perp \overline{BC}$.

Prove: Ray n contains point X, and $XD = XE = XF$.



25. Noise Control You are trying to talk to a friend on the phone in a busy bus station. The buses are so loud that you can hardly hear. Referring to the figure at the right, should you stand at P or C to be as far as possible from all the buses? Explain.



Reasoning **Determine whether each statement is *true* or *false*. If the statement is false, give a counterexample.**

26. The incenter of a triangle is equidistant from all three vertices.

27. The incenter of a triangle always lies inside the triangle.

28. You can circumscribe a circle about any three points in a plane.

29. If point C is the circumcenter of $\triangle PQR$ and the circumcenter of $\triangle PQS$, then R and S must be the same point.

Challenge

30. **Reasoning** Explain why the circumcenter of a right triangle is on one of the triangle's sides.

Determine whether each statement is *always*, *sometimes*, or *never* true. Explain.

31. It is possible to find a point equidistant from three parallel lines in a plane.

32. The circles inscribed in and circumscribed about an isosceles triangle have the same center.

Standardized Test Prep

SAT/ACT

33. Which of the following statements is *false*?

(A) The bisectors of the angles of a triangle are concurrent.

(B) The midsegments of a triangle are concurrent.

(C) The perpendicular bisectors of the sides of a triangle are concurrent.

(D) Four lines intersecting in one point are concurrent.

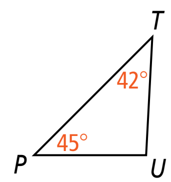
34. What type of triangle is $\triangle PUT$?

(F) right isosceles

(G) acute isosceles

(H) obtuse scalene

(I) acute scalene



35. Which statement is logically equivalent to the following statement?

If a triangle is right isosceles, then it has exactly two acute angles.

(A) If a triangle is right isosceles, then it has one right angle.

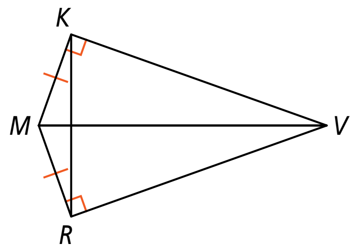
(B) If a triangle has exactly two acute angles, then it is right isosceles.

(C) If a triangle does not have exactly two acute angles, then it is not right isosceles.

(D) If a triangle is not right isosceles, then it does not have a right angle.

Short Response

36. Refer to the figure at the right. Explain in two different ways why $\overline{MV}$ is the angle bisector of $\angle KVR$.

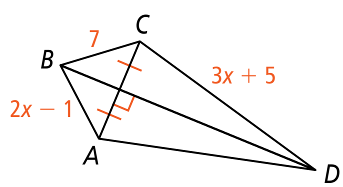


Mixed Review

Use the figure at the right for Exercises 37 and 38.

37. Find the value of x.

38. Find the length of $\overline{AD}$.



See Lesson 5-2.

Get Ready! **To prepare for Lesson 5-4, do Exercises 39 and 40.**

Find the coordinates of the midpoint of $\overline{AB}$ with the given endpoints.

See Lesson 1-7.

39. $A(3, 0)$, $B(3, 16)$

40. $A(6, 8)$, $B(4, -1)$